arXiv:hep-th/0512217v1 17 Dec 2005

hep-th/0512217
MAD-TH-05-08

Open/closed duality for FZZT branes in $c = 1$

Ian Ellwood and Akikazu Hashimoto

Department of Physics
University of Wisconsin
Madison, WI 53706

Abstract

We describe how the matrix integral of Imbimbo and Mukhi arises from a limit of the FZZT partition function in the double-scaled $c = 1$ matrix model. We show a similar result for 0A and comment on subtleties in 0B.

1 Introduction

Non-critical string theories with $c \leq 1$ are an important conceptual laboratory for exploring the inner workings of string theory. As world sheet theories, they are formulated by coupling $c \leq 1$ matter to Liouville theory and describe strings propagating in a space-time with a linear dilaton and a tachyon condensate. While these world sheet theories are, in general, as difficult to solve as critical string theories, it is a special feature of low-dimensional strings that there often exists a dual description in terms of an exactly solvable matrix model [1–5].

There are, typically, *two* distinct formulations of a given $c \leq 1$ string as a matrix model: the double-scaled matrix model and the Kontsevich model. Historically, these models were derived in completely different ways. The double-scaled matrix model is based on 't Hooft's idea that, at large N, the Feynman diagrams of gauge theory can be thought of as random triangulations of a string world sheet. The Kontsevich model, meanwhile, is found by directly solving the flow equations implied by the integrable structure. Amazingly, this solution takes the form of a simple matrix integral [6]. Both approaches can be used to compute observables, such as correlation functions of vertex operators on the world sheet, and give identical results. However, until recently, it was not understood how they are related to each other.

The existence of dual formulations, one in terms of a string theory and the other in terms of matrices, is reminiscent of open/closed-string duality and suggests that the matrix models may arise from open-string theories living on branes. There are two classes of branes in low dimensional string theories, which arise from two possible boundary conditions in the Liouville sector: FZZT branes [7,8], which are the analogue of Neumann boundary conditions and are extended, and ZZ branes [9], which are the analogue of Dirichlet boundary conditions and are localized in the strong coupling region.

ZZ branes are unstable. This, together with the fact that they live in the strongly coupled region, makes a direct construction of the open string theory living on them difficult. However, it was shown in [10, 11] that the decay of a ZZ brane can be interpreted as a single eigenvalue of the double-scaled matrix model rolling down the unstable potential. This allows one to think of the double-scaled matrix model as the open string theory living on an infinite number of ZZ-branes that have condensed to form a Fermi-surface.

FZZT branes are stable. They are labeled by a parameter μ_B, known as the boundary cosmological constant. Although the FZZT brane is the analogue of Neumann boundary conditions, it is only semi-infinite in extent. For example, if we denote by $|\mu_B\rangle$ the boundary state of an FZZT-brane in the (p, q) minimal model, then, in the mini-superspace approxi-

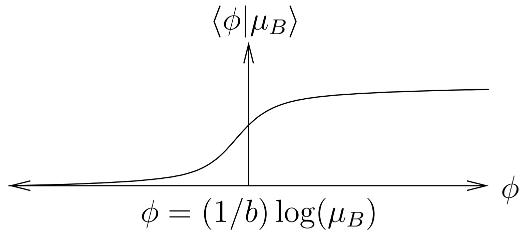

Figure 1: Illustration of profile $\langle\phi|\mu_B\rangle \sim e^{-\mu_B e^{-b\phi}}$ of the boundary state $|\mu_B\rangle$

mation, we have the profile[1],

$$\langle\phi|\mu_B\rangle \sim e^{-\mu_B e^{-b\phi}}, \qquad \text{where} \qquad b^2 = \frac{p}{q}, \qquad c = 1 - \frac{6(p-q)^2}{pq} \ . \tag{1.1}$$

As one can see in figure 1, the brane exists throughout the weak coupling region at large ϕ, but disappears around $\phi \sim \frac{1}{b}\log\mu_B$. Hence, by varying μ_B, FZZT branes can be used as a probe of target-space [12].

Having identified the open string theory on ZZ branes with the double-scaled matrix model, it is natural to try to identify the Kontsevich model with open strings living on FZZT branes. The precise relation was first pointed out by Gaiotto and Rastelli who considered Witten's cubic open string field theory [13] for the open strings ending on stack of n FZZT branes for the case of the minimal $(p,q) = (2,1)$ model [14]. Remarkably, they found that the open string field theory [13] on a stack of FZZT branes could be reduced to precisely the action of the $n \times n$ Kontsevich matrix integral[2],

$$S(M,\Lambda) = \mathrm{Tr} M\Lambda - \frac{M^3}{3} \ . \tag{1.2}$$

Unfortunately, while this result is elegant, it has not been easy to generalize it to other examples, such as the $(p,q) \neq (2,1)$ models [14, 15].

An alternate approach, which seems to generalize more easily, is to compute the open string theory on the FZZT branes using the double-scaled matrix model. An FZZT brane can be incorporated into the double-scaled matrix model by inserting into the path integral the exponential of the loop operator,

$$\exp(\mathrm{Tr}\log(\mu_B - \Phi)) = \det(\mu_B - \Phi), \tag{1.3}$$

where Φ is the $N \times N$ matrix of the double-scaled matrix model. This approach was pursued in [16], and it was shown that one could derive the Kontsevich model from the partition function of the (2,1) matrix model in the presence of a stack FZZT branes.

[1] Our convention is that $\phi \to +\infty$ is the weakly coupled region.

[2] Sometimes, the action is presented in a form where the term linear in M is replaced by term quadratic in M by shifting M.

Unlike the string field theory analysis, the double-scaled matrix model analysis can be generalized in a straightforward fashion to other examples. For example, it was shown in [17] that, using the the double-scaled $(p, 1)$ model [18] in the presence of n FZZT-branes, one could derive the corresponding generalization of the Kontsevich integral,

$$Z(\Lambda) = \int dM \, e^{-\mathrm{Tr}\left(M\Lambda - \frac{M^{p+1}}{p+1}\right)} \,. \tag{1.4}$$

In spite of the success of these arguments in $c < 1$ models, the extension to $c = 1$ has not been worked out. As with the $c < 1$ models there is an analogue of Kontsevich matrix integral,

$$Z_n(A, \bar{t}) = (\det A)^\nu \int dM \, e^{\mathrm{Tr}\left(-\nu MA + (\nu - n)\log M - \nu \sum_{k=1}^{\infty} \bar{t}_k M^k\right)}, \qquad \nu = -i\mu \,, \tag{1.5}$$

first derived by Imbimbo and Mukhi [19]. However, the connection between the open string theory on FZZT-branes and the Imbimbo-Mukhi model has not been made.

Several attempts to make this connection can be found in the literature. One approach is to work out the string field theory living on a stack of FZZT-branes in the same spirit as the Gaiotto-Rastelli computation. It is not clear, however, how the degrees of freedom of open strings ending on a semi-infinite world volume should arrange themselves into parameters A and $\bar{t}$ of the Imbimbo-Mukhi model and the authors of [20] did not succeed in recovering the Imbimbo-Mukhi model this way.

Another method of deriving the Imbimbo-Mukhi model used the so-called normal-matrix model [21]. This model is similar to the Imbimbo-Mukhi model in that it is a matrix integral solution to the integrable flow equations. As discussed in [22], one can perform a computation similar to [16, 17] in which one evaluates the expectation value of certain determinant operators in the model to reproduce the Imbimbo-Mukhi model. While this gives a nice connection between the normal-matrix model and the Imbimbo-Mukhi model, the relationship between the determinant operators in the normal-matrix model and FZZT branes is not clear.

2 FZZT branes in bosonic $c = 1$ string theory

Having reviewed what is known about the $c = 1$ case, we can now turn to the main goal of this paper: we wish to fill in the conceptual gap between the open string theory on the FZZT branes and the Imbimbo-Mukhi model. To do so, we attempt to compute the expectation value of the exponential of the loop operator, as was done in the $c < 1$ case, and manipulate it into the form of the Imbimbo-Mukhi matrix integral [17]. As we will see, many of the

ingredients that were used in the $c<1$ computation have analogues in the $c=1$ case. There is, however, one new subtlety. As described below, one must take a certain scaling limit of the FZZT partition function before one can derive the Imbimbo-Mukhi model.

We begin our discussion with a brief review of the relevant aspects of the $c=1$ matrix model that we need for our derivation. We mostly follow the approach of Kostov [23, 24]. The matrix-model action is given by

$$S = \mathrm{Tr} \int_0^\beta dx \left(-\tfrac{1}{2}(\nabla_A \Phi)^2 + \tfrac{1}{2}\Phi^2 \right), \tag{2.1}$$

where

$$\nabla_A \Phi = \partial_x \Phi - i[A, \Phi], \tag{2.2}$$

and $\Phi(x)$ and $A(x)$ are $N \times N$ Hermitian matrices. For the bosonic theory, we work in units where $\alpha' = 1$. Since we are interested in deriving the Imbimbo-Mukhi model, we work with the Euclidean action and compactify the time direction with period $\beta = 2\pi R$.

It is useful to rewrite the action, (2.1), in chiral form. To do this, one first switches to the first order form,

$$S = \mathrm{Tr} \int_0^\beta dx \left(iP(\nabla_A \Phi) - \tfrac{1}{2}P^2 + \tfrac{1}{2}\Phi^2 \right), \tag{2.3}$$

which reduces to equation (2.1) upon integrating out P. Then, substituting

$$X_\pm = \frac{\Phi \pm P}{\sqrt{2}}, \tag{2.4}$$

one arrives at the chiral action,

$$S = \mathrm{Tr} \int_0^\beta dx (-iX_+ \nabla_A X_- + X_+ X_-). \tag{2.5}$$

In this form it is relatively straightforward to reduce the path integral over $X_\pm$ and A to an ordinary three-matrix integral [23–25]. Defining $X_\pm = X_\pm(t=0)$ one finds,

$$Z_N = \int dX_+ dX_- d\Omega e^{i\mathrm{Tr}(X_+X_- - q^{-1}X_+\Omega X_-\Omega^{-1})}, \tag{2.6}$$

where the subscript, N, is a reminder that the partition function depends on the rank, N, of the matrices and $q = e^{i\beta}$. The matrix, $\Omega \in U(N)$, is given by

$$\Omega = P\left[e^{i\int_0^\beta A(x)dx} \right]. \tag{2.7}$$

This expression can be simplified further by integrating out Ω, which reduces the partition function to an integral over the eigenvalues of $X_\pm$,

$$Z_N = \int \prod_{k=1}^N dx_k^+ dx_k^- \det_{jk}(e^{-ix_j^- x_k^+}) \det_{jk}(e^{iq^{-1}x_j^- x_k^+}). \tag{2.8}$$

In order to extract the observables of the $c = 1$ matrix model, one computes the grand partition function, which, following [24, 25], can be expressed in the form of a Fredholm determinant,

$$Z(\mu) = e^{-\beta\mu N} Z_N = \det(1 + e^{-\beta\mu} K_- K_+), \tag{2.9}$$

where

$$\begin{aligned} [K_+ f](x_-) &= \int dx_+ e^{iq^{-1}x_+ x_-} f(x_+), \\ [K_- f](x_+) &= \int dx_- e^{-ix_+ x_-} f(x_-) \end{aligned} \tag{2.10}$$

are operators acting on functions of x_+ and x_-, respectively. The Fredholm determinant is computed by diagonalizing the operators K_+ and K_- and taking the product of their eigenvalues. This closely resembles what one does in computing the partition for the two-matrix models in terms of orthogonal polynomials. Up to non-perturbative corrections, the partition function of the undeformed Euclidean $c = 1$ model can be written as

$$\log Z(\mu) = \sum_{r=Z+\frac{1}{2}} i\phi_0(-\mu + \frac{i}{R} r), \qquad e^{i\phi_0(E)} = e^{i\pi/4} \frac{\cosh(\pi E)}{\pi} \Gamma\left(\frac{1}{2} + iE\right). \tag{2.11}$$

As is well known, when compactified on a circle, the $c = 1$ theory contains a tower of tachyon deformations with discrete Euclidean momenta. These deformations change the asymptotic shape of the Fermi surface in phase space. One can incorporate them into the matrix model by generalizing (2.8) to

$$Z_N = \int \prod_{k=1}^{N} [dx_k^+][dx_k^-] \det_{jk}(e^{-ix_j^- x_k^+}) \det_{jk}(e^{iq^{-1}x_j^- x_k^+}), \tag{2.12}$$

where

$$[dx_\pm] = e^{\pm iU_\pm(x_\pm)} dx_\pm, \qquad U_\pm(x_\pm) = iz_{\pm k} x_\pm^{k/R} \tag{2.13}$$

and the z_k are the deformation parameters corresponding to tachyons with momentum k. The sign and the phase of z_k was chosen so that it coincides with the deformation parameter of [26]. This identification is explained in more detail in appendix A. The grand partition function is still given by the Fredholm determinant, but for the deformed operator,

$$Z(\mu) = \det(1 + e^{\beta\mu} \tilde{K}_- \tilde{K}_+), \tag{2.14}$$

where

$$\begin{aligned} [\tilde{K}_+ f](x_-) &= \int [dx_+] e^{iq^{-1}x_+ x_-} f(x_+), \\ [\tilde{K}_- f](x_+) &= \int [dx_-] e^{-ix_+ x_-} f(x_-) \ . \end{aligned} \tag{2.15}$$

It was shown in [23] that such a deformation can be described in terms of a Toda flow and that the partition function is related to the τ-function via[3]

$$Z(\mu) = \tau\left(R(i\mu - \tfrac{1}{2}) + \tfrac{1}{2}, z_+, z_-\right) \ . \tag{2.16}$$

The initial condition for the flow, (2.11), can also be expressed in the form of the string equation, found in [30, 31].

Having given a brief review of the $c = 1$ matrix model, we now study what happens when we add an FZZT-brane to the story. As mentioned earlier, an FZZT-brane can be incorporated in the matrix model by inserting into the path integral an exponential of the macroscopic-loop operator,

$$e^{\pm W(x)} = \det(\Phi(x) - \mu_B)^{\pm 1} = \det\Big((X_+(x) + X_-(x))/\sqrt{2} - \mu_B\Big)^{\pm 1}, \tag{2.17}$$

as first considered in [32]. Note that the determinant operator generates an FZZT brane with Dirichlet boundary conditions in the $c = 1$ direction and, hence, is localized in Euclidean time at a point x. The "$\pm$" in the exponent is included so that we can simultaneously consider the determinant and the inverse determinant operator insertions. As we will see later, the connection with the Imbimbo-Mukhi model favors the inverse determinant [22]. However, inserting the determinant also gives rise to a matrix integral that encodes the τ-function of the Toda integrable flow and is, roughly, the complex conjugate of the Imbimbo-Mukhi model.

Inserting (2.17) into the chiral Lagrangian, (2.5), one sees that the determinant has a simple x dependence,

$$e^{\pm W(x)} = \det\Big((e^{ix/R}X_+(t=0) + e^{-ix/R}X_-(t=0))/\sqrt{2} - \mu_B\Big)^{\pm}. \tag{2.18}$$

One can also see this x dependence by considering the Heisenberg form of the operators, $X_\pm$, in the Hamiltonian language.

Now, it would be nice if inserting this operator into the path integral was equivalent to a tachyon deformation. It clearly is not, since the determinant contains a complicated mixture of X_+ and X_- operators, and does not factorize into an X_+ term and an X_- term in the way that the tachyon deformation, (2.13), does. However, as we now show, it is possible to get a pure tachyon deformation if we take a certain limit of the determinant operator.

Suppose that we analytically continue $x = -it$ and send $t \to \infty$ while simultaneously rescaling

$$\mu_B = e^t \mu'_B \ . \tag{2.19}$$

[3]The connection between $c = 1$ matrix model and the Toda integrable hierarchy was first pointed out in [27]. For introduction to Toda lattice hierarchy, see [26, 28, 29]

Then, up to an overall normalization factor, the operator asymptotes to

$$\det\left(1-\frac{X_+}{\mu'_B}\right)^{\pm 1} = e^{\mp\sum_n \frac{1}{n}\mu_B'^{-n}X_+^n} \ . \tag{2.20}$$

In the case where the radius of the $c = 1$ direction is one, inserting this operator into the partition function is equivalent to a shift in the Toda flow,

$$\delta z_k = \pm\frac{1}{n}\mu_B'^{-n}, \qquad \delta k > 0. \tag{2.21}$$

Readers familiar with [22] will recognize this deformation as the same one generated by one of the determinants in the normal-matrix model, $\det(1 - Z/\mu_B)^{\pm 1}$. The other determinant, $\det(1 - Z^\dagger/\mu_B)^{\pm 1}$, can generated by taking the opposite sign in the analytic continuation, $x \to it$. Notice that away from $R = 1$ there does not seem to be any relation between the FZZT-brane determinant and the determinant of the normal matrix model.

Once we have identified that our limit of the FZZT determinant generates the Toda flow, (2.21), the integrable structure of the model is enough to determine that the expectation value of (2.20) will give the Imbimbo-Mukhi model, as was shown in [22] using the normal-matrix model. However, we would like to understand how to derive the Imbimbo-Mukhi model more directly.

As in the $c < 1$ case, it is useful to introduce the Baker-Akhiezer function, $\Psi_s^\pm(\lambda)$, which is given in terms of the τ-function by

$$\begin{aligned}
\Psi_s^+(\lambda) &= \frac{\tau(z_+ - \epsilon(\lambda^{-1/R}), z_-, s)}{\tau(z_+, z_-, s)}\exp[z_+(\lambda^{1/R}) + s\log(\lambda^{1/R})], \\
\Psi_s^-(\lambda) &= \frac{\tau(z_+, z_- - \epsilon(\lambda^{1/R}), s+1)}{\tau(z_+, z_-, s)}\exp[z_-(\lambda^{-1/R}) + s\log(\lambda^{1/R})],
\end{aligned} \tag{2.22}$$

where

$$\epsilon(\lambda) = \left(\lambda, \frac{\lambda^2}{2}, \frac{\lambda^3}{3}, \ldots\right), \qquad z_+(\lambda) = \sum_{n=1}^{\infty} z_n\lambda^n, \qquad z_-(\lambda^{-1}) = \sum_{n=1}^{\infty} z_{-n}\lambda^{-n} \ . \tag{2.23}$$

For a formal definition of the Baker-Akhiezer function in terms of dressing operators in the Lax formalism, see equations (25) and (27) of [33].

Notice that, in the expression for Ψ_s^+, the argument of the τ-function is shifted by $z_+ \to z_+ - \epsilon(\lambda^{-1/R})$. Setting $\lambda = \mu'_B$ and $R = 1$, we see that this is the same shift as (2.21), provided we consider the *inverse* determinant. Hence, the τ-function in the presence of the inverse determinant, (2.20), can be computed if we know the Baker-Akhiezer function.

By itself, this fact is not very useful. However, the Baker-Akhiezer functions are also closely related to the biorthogonal eigenfunctions, $\psi_\pm$, of the deformed operators, $\tilde{K}_\pm$ given

in (2.15) [23]:

$$\psi_+^E(x_+)^* = \Psi_s^+(x_+)\big|_{s=-R(iE+1/2)}\,, \qquad \psi_-^E(x_-)^* = x_-\Psi_s^-(x_-^{-1})\big|_{s=-R(iE+1/2)}\,. \tag{2.24}$$

Since the biorthogonal eigenfunctions are related to each other by Fourier transform, it follows that Ψ_s^+ and Ψ_s^- are related by

$$\Psi_s^+(\lambda_+) = \int d\lambda_-^{-1} e^{i\lambda_+\lambda_-^{-1}} \lambda_- \Psi_s^-(\lambda_-). \tag{2.25}$$

Now, suppose that $z_+ = 0$. Then the non-trivial pieces of Ψ_s^- drop out and Ψ_s^- is completely determined as a function of z_- and λ;

$$\Psi_s^-(\lambda_-) = \mathcal{N} \exp[z_-(\lambda_-^{-1/R}) + s\log(\lambda_-^{1/R})], \tag{2.26}$$

where $\mathcal{N}$ is a normalization factor independent of λ and z_-, which we can ignore. Changing variables to $m = \lambda_-^{-1}$, one has

$$\Psi_s^+(a) = \int dm\, \exp[iam + z_-(m^{1/R}) - (s+R)\log(m^{1/R})]. \tag{2.27}$$

Using (2.22), one finds

$$\tau(-\epsilon(a^{-1/R}), z_-, s) = \int dm\, (ma)^{-s/R} \exp[iam + z_-(m^{1/R}) - \log(m)], \tag{2.28}$$

which is equivalent to

$$Z_1(A, t_-) = \tau(-\epsilon(a^{-1/R}), z_-, s) = \int dM\, (MA)^{\nu R + (R-1)/2} e^{-\nu((MA)^R + t_-(M)) - \log(M)}, \tag{2.29}$$

with the identifications,

$$t_{-n} = i\mu^{n/2R-1} z_{-n}, \qquad m = \mu^{1/2} M^R, \qquad a = \mu^{1/2} A^R, \qquad s = i\mu R - \frac{R-1}{2}, \qquad \nu = -i\mu. \tag{2.30}$$

As we discuss in appendix C, this argument can be iterated to give the general expression,

$$Z_n(A, t_-) = \int dM_i \frac{\Delta(M)}{\Delta(A)} \prod_{i=1}^{n} \left[(M_i A_i)^{\nu R + (R-1)/2} e^{-\nu((M_i A_i)^R + \frac{1}{n} t_{-n} M_i^n) - n\log(M_i)} \right]\,, \tag{2.31}$$

with Miwa-Kontsevich-like transform,

$$t_n = -\frac{1}{\nu k} A^{-k}\,. \tag{2.32}$$

This is the same result obtained in [22] using the normal-matrix model.[4] Setting $R = 1$, we reproduce precisely the Imbimbo-Mukhi integral.

[4]It should be emphasized that while the sign of z_n's is fixed by matching to the conventions of [26], there is no intrinsic definition for the sign of the t_n's. In particular, the sign convention for the t_n's in [19] and [22] appear to differ by a sign.

Recall that, to match with the argument of the Baker-Akhiezer function, we had to consider the inverse determinant. This is in agreement with the observation made in [22] that the Imbimbo-Mukhi integral is derived from the insertion of an inverse determinant in the normal-matrix model.

One can also consider the insertion of a determinant. In this case, one must work with the dual Baker-Akhiezer functions [29][5],

$$\begin{aligned} \Psi_s^{+*}(\lambda) &= \frac{\tau(z_+ + \epsilon(\lambda^{-1/R}), z_-, s)}{\tau(z_+, z_-, s)} \exp[-z_+(\lambda^{1/R}) - s\log(\lambda^{1/R})], \\ \Psi_s^{-*}(\lambda) &= \frac{\tau(z_+, z_- + \epsilon(\lambda^{1/R}), s-1)}{\tau(z_+, z_-, s)} \exp[-z_-(\lambda^{-1/R}) - s\log(\lambda^{1/R})] \ . \end{aligned} \tag{2.33}$$

As with the Baker-Akhiezer functions, they are related by Fourier transform;

$$\frac{1}{\sqrt{2\pi}}\Psi_s^{+*}(\lambda_+) = \lambda_+^{1-1/R}\int d\lambda_-^{-1}\, e^{-i\lambda_+\lambda_-^{-1}}\frac{1}{\sqrt{2\pi}}\lambda_-^{1/R}\Psi_s^{-*}(\lambda_-), \tag{2.34}$$

from which one can derive,

$$\begin{aligned} Z_n(A,t_-) &= \int dM_i \frac{\Delta(M)}{\Delta(A)}\prod_{i=1}^{n}\left[(M_iA_i)^{\nu R+(R-1)/2}e^{-\nu((M_iA_i)^R+\frac{1}{n}t_{-n}M_i^n)-n\log(M_i)}\right] \ , \\ t_n &= -\frac{1}{\nu k}A^{-k}, \qquad \nu = i\mu \ . \end{aligned} \tag{2.35}$$

For $R=1$, one can use the Harish-Chandra/Itzykson-Zuber formula [34, 35] to re-express this integral in matrix form. This again yields a matrix model identical to the Imbimbo-Mukhi model. The only difference between (2.35) and (2.31) is that we have changed the relative sign between ν and $i\mu$. Since both models are an even function of μ, this does not have any effect on the closed string observables. Therefore, at the level of reproducing the τ-function, one can not distinguish between the determinant and the inverse determinant. This choice is ultimately tied to whether the FZZT-ZZ strings are bosons or fermions.

When $R \neq 1$ it is still true that (2.31) and (2.35) give complete information about the τ-function. However, since the FZZT determinants no longer give rise to a Toda flow, (2.31) and (2.35) are no longer related – in any obvious way – to FZZT-branes. Thus, we find that the open/closed string duality, relating the Kontsevich-like matrix integral and the FZZT brane, appears to work only at self dual radius. This is consistent with the observation in [22] that (2.31) and (2.35) cannot be represented in the form of a matrix integral except at $R=1$. After all, if (2.31) and (2.35) were derivable from an open/closed duality, one would expect the result to take the form of a matrix integral for general R.

[5] They are related to $w^{\binom{\infty}{0}*}(s, z_+, z_-, \lambda)$ in the notation of [26] according to

$$\Psi_s^{+*}(\lambda) = w^{(\infty)*}(s,z_+,z_-,\lambda)e^{-z_+(\lambda^{1/R})}\lambda^{-s/R}, \qquad \Psi_s^{-*}(\lambda) = w^{(0)*}(s,z_+,z_-,\lambda)e^{-z_-(\lambda^{1/R})}\lambda^{-s/R} \ .$$

3 FZZT branes in $\hat{c} = 1$ string theories

Our analysis relating the determinant operator to a Kontsevich-like matrix integral can be extended to the $\hat{c} = 1$ models [36–38]. There are two $\hat{c} = 1$ models: type 0A and type 0B. We consider them separately.

3.1 Type 0A

The type 0A theory is formulated in terms of a $U(N) \times U(N+q)$ gauged matrix model [37,38]. The parameter q gives the number of units of Ramond-Ramond flux in the background.[6] As in the bosonic case, one can fix the gauge and reduce the matrix degrees of freedom to the eigenvalues. This gives rise to a model which is equivalent to the deformed matrix model originally constructed in [40]. We will work in units where $\alpha' = 1/2$ so that the harmonic oscillator has the same oscillator frequency as the $c = 1$ model. Physical observables such as the S-matrix and the partition function of this model were computed in [41]. These computations were then reformulated in light-cone variables [39], which are better suited for the analyzing the deformations and Toda integrable structure of the model [42].[7]

The main difference between the type 0A model and the bosonic $c = 1$ model is that, in 0A, the wave functions, $\psi_+^E(x_+)$ and $\psi_-^E(x_-)$, are related by a Bessel transform instead of a Fourier transform. As a result, the Baker-Akhiezer functions are also related by an integral Bessel transform,

$$\Psi_s^+(\lambda_+) = \int_0^\infty d\lambda_-^{-1} \sqrt{\lambda_+ \lambda_-^{-1}} J_q(\lambda_+ \lambda_-^{-1}) \lambda_- \Psi_s^-(\lambda_-). \tag{3.1}$$

The partition function can be written in the form[8],

$$Z_N = \int \prod_{k=1}^N [dx_k^+][dx_k^-] \det_{jk} \left(\sqrt{x_j^- x_k^+} J_q(x_j^- x_k^+) \right) \det_{jk} \left(\sqrt{x_j^- x_k^+} J_q(e^{-i\beta} x_j^- x_k^+) \right), \tag{3.2}$$

where

$$[dx_\pm] = e^{\pm i U_\pm(x_\pm)} dx_\pm, \qquad U_\pm(x_\pm) = i z_{\pm k} x_\pm^{k/R}. \tag{3.3}$$

We can also express $Z(\mu)$ as a Fredholm determinant,

$$Z(\mu) = \det(1 + e^{\beta\mu} \tilde{K}_- \tilde{K}_+), \tag{3.4}$$

[6] See [39] for a discussion of the subtleties associated with the two ways of introducing charges in this model.

[7] See also [43–46] for a related discussion in the context of minimal models.

[8] Note that q here refers to the background flux and not $e^{i\beta}$.

where

$$[\tilde{K}_+ f](x_-) = \int [dx_+] \sqrt{x_+ x_-} J_q(e^{-i\beta} x_+ x_-) f(x_+) \ ,$$
$$[\tilde{K}_- f](x_+) = \int [dx_-] \sqrt{x_+ x_-} J_q(-x_+ x_-) f(x_-) \ . \tag{3.5}$$

Using the relation between the τ-function and the Baker-Akhiezer function, (2.22), one finds

$$\tau(\mp\epsilon(a^{-1/R}), z_-, s) = \int dm (ma)^{-s/R+1/2} J_q(am) \exp[\pm z_-(m^{1/R}) - \log(m)], \tag{3.6}$$

which generalizes (2.28) to type 0A. Just as in the $c = 1$ case, one can rescale

$$t_{-n} = i\mu^{n/2R-1} z_{-n}, \qquad m = \mu^{1/2} M^R, \qquad a = \mu^{1/2} A^R \tag{3.7}$$

to bring this expression into an Imbimbo-Mukhi form. Generalizing to the multiple FZZT case one finds the expression,

$$\tau(\mp\epsilon(a^{-1/R}), z_-, s)$$
$$= \int dM \, \frac{\Delta(M)}{\Delta(A)} \prod_{i=1}^{n} (M_i A_i)^{-s+R/2} J_q(\pm\mu A_i^R M_i^R) \exp[\pm i\mu(t_-(M_i)) - n\log(M_i)], \tag{3.8}$$

where

$$s = \pm i\mu R - \frac{R-1}{2} \ . \tag{3.9}$$

Using

$$\int d\theta \, e^{iq\theta - \nu ma\cos\theta} = J_q(m) \ , \tag{3.10}$$

one can integrate in an additional variable and express the shifted τ-function in a two-integral form:

$$\tau(\mp\epsilon(a^{-1/R}), z_-, s) = \frac{1}{\Delta(z)^{1/2R}} \prod_{i=1}^{n} \left(z_i^{\nu/2+(R-1/2)/4R+q/2} \right) \int_0^\infty ds_i \prod_{i=1}^{n} \left(\frac{1}{s_i} e^{-s_i z_i + q} \right)$$
$$\times \int dy_i \, \Delta(y)^{1/2R} \prod_{i=1}^{n} y_i^{\nu/2+(R-1/2)/4R-(n+q/2)} e^{-y_i/s_i} \exp[-\nu(t_-(y_i^{1/2R}))] \ , \tag{3.11}$$

where we have used the change of variables,

$$y_i = m_i^2 \ , \qquad s = \frac{m}{a\nu}\cos\theta \ , \qquad z_i = \nu^2 a_i^2 \ , \qquad \nu = \mp i\mu \ . \tag{3.12}$$

Setting $R = 1/2$, one recovers a matrix integral representation found in [47, 48].

To relate the Baker-Akhiezer function and the τ-function to FZZT branes, we consider the determinant operator [38],

$$e^{\pm W(x)} = \det(X(x)X(x)^\dagger - \mu_B)^{\pm 1} = \det\Big((X_+(x) + X_-(x))(X_+^\dagger(x) + X_-^\dagger(x))/2 - \mu_B\Big)^{\pm 1} . \tag{3.13}$$

Taking the same scaling limit as we did in the bosonic case, (3.13) reduces to

$$\det\left(1 - \frac{X_+X_+^\dagger}{\mu_B'}\right)^{\pm 1} = e^{\mp \sum_n \frac{1}{n}\mu_B'^{-n}(X_+X_+^\dagger)^n} . \tag{3.14}$$

Comparing this with the parameterization of the deformation parameter, (2.13), we see that the exponentiated macroscopic-loop operator matches with the Baker-Akhiezer function at $R = 1/2$. This provides the link between FZZT branes and the Imbimbo-Mukhi integral, (3.8), as well as the matrix integral of [47, 48].

Since $\hat{c} = 1$ models are on firmer non-perturbative footing that the bosonic $c = 1$ models, it is instructive to examine the the deformed τ-function function at the non-perturbative level. To that end, it is useful to go back to the integral transform, (3.1), and write it in the form,

$$\Psi_+^s(\lambda_+) = \int_0^\infty d\lambda_-^{-1}\sqrt{\lambda_+\lambda_-^{-1}}\left(H_q^+(\lambda_+\lambda_-^{-1}) + H_q^-(\lambda_+\lambda_-^{-1})\right)\Psi_-^s(\lambda_-), \tag{3.15}$$

where

$$H_q^\pm(x) = \frac{J_q(x_+x_-) \pm iY_q(x_+x_-)}{2} \tag{3.16}$$

are the Henkel functions. Assuming that the background is only deformed by z_-, one can use the relation between the Baker-Akhiezer function and the τ-function to write

$$\tau(-\epsilon(a^{-1/R}), z_-, s) = \int_0^\infty dm(ma)^{-s/R+1/2}(H_q^+(am) + H_q^-(am))\exp[z_-(m^{1/R}) - \log(m)] . \tag{3.17}$$

To define the theory non-perturbatively, one must specify the contour of integration for the variable, m. For the undeformed background, $z_- = 0$, one can integrate the Henkel function $H_q^\pm(x)$ anywhere along the upper/lower half of the complex x plane. This choice of contour reproduces the correct perturbative expansions in z_-.

When the deformation parameters z_- are finite, one encounters a discrete choice of allowed contours that give rise to finite but distinct results. To illustrate this point, consider the simplest case of $R = 1/2$ with z_{-1} being the only non-vanishing deformation parameter. We will also take μ to be real and positive, and t_{-1} to be real but have small negative imaginary part. Then, z is mostly imaginary with a small negative real part. The asymptotic behavior of the integrand (3.17) is dominated by the $\exp(z_{-1}m^2)$ factor and converges when integrated along contours that go to infinity along the unshaded region, as illustrated in figure 2.

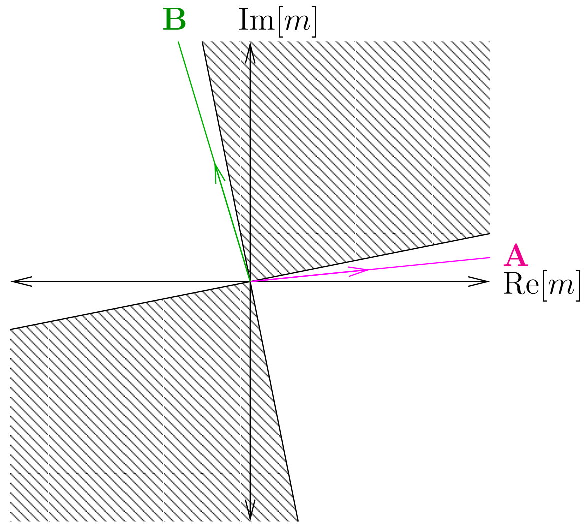

Figure 2: An illustration of possible contours of integration in m-space for which the integral (3.17) converges. Contours asymptoting to infinity along the shaded regions are excluded. Here it is assumed that z_{-1} is the only non-vanishing deformation parameter.

The integral involving $H_q^+(am)$ can be performed along either of the contours labeled by **A** and **B** in the figure. The difference between the two contours will only contribute non-perturbatively in the asymptotic expansion in small $t_{-1}/a^2 = t_{-1}/\mu^2 A^2$. For a more general deformation, there could be many distinct contours consistent with the perturbative expansion. Therefore, in order to define the τ function non-perturbatively, we need a definite prescription for the contour.

One natural requirement is that the contour have a well defined limit as the $t_{-n} \to 0$. Let us assume, for simplicity, that there is some $k > 0$ such that $t_{-n} = 0$ for $n > k$.[9] Then, the factor of $\exp[z_-(m^{1/R})]$ is always oscillating along the positive real axis. By adding a small imaginary part with appropriate sign to t_{-k}, one can open a small region around the positive axis for which the exponential factor $\exp[z_-(m^{1/R})]$ converges. This is illustrated in figure 2. Both Henkel functions can then be integrated over contours on the correct side of – and arbitrarily close to – the real axis. In figure 2, this corresponds to choosing the contour **A**. We propose that this contour is the correct non-perturbative definition of our integral, (3.17). This prescription gives a natural, though not unique, non-perturbative definition of the Toda flow. It would be instructive if one could find some consistency check for this prescription.

[9] Such deformations were also discussed in [49].

3.2 Type 0B

Finally, let us briefly describe the extension to the case of type 0B theory. This model turns out to be more subtle at the non-perturbative level than the type 0A model.

We first consider the case of vanishing background RR flux. The model is formulated as the quantum mechanics of an inverted harmonic oscillator potential with fermions filling both sides of the potential to the same level. There is an explicit Z_2 symmetry exchanging left and the right side of the potential, and, as long as the unbounded potential is regularized in a way that preserves the left/right symmetry, the spectrum of states can be separated into sectors which are odd and even with respect to this Z_2. Since these two sectors are decoupled from one another, the partition function factorizes into a product of two τ-functions [42],

$$Z(\mu) = \tau_{odd}(z_{odd+}, z_{odd-}, s)\tau_{even}(z_{even+}, z_{even-}, s). \tag{3.18}$$

The Baker-Akhiezer functions for the odd and even sectors have an integral transform,

$$\begin{aligned}
\Psi^s_{odd+}(\lambda_+) &= \int_0^\infty d\lambda_-^{-1} \sin(\lambda_+\lambda_-^{-1})\Psi^s_{odd-}(\lambda_-),\\
\Psi^s_{even+}(\lambda_+) &= \int_0^\infty d\lambda_-^{-1} \cos(\lambda_+\lambda_-^{-1})\Psi^s_{even-}(\lambda_-).
\end{aligned} \tag{3.19}$$

Using the relation between τ-functions and the Baker-Akhiezer functions, one arrives at an integral expression of the form,

$$\begin{aligned}
&\tau_{even}(t_k, t_{-k}, s) = \\
&\quad \int_0^\infty dM_i \, \frac{\Delta(M)}{\Delta(A)} \prod_{i=1}^n (M_i A_i)^{\nu R + (R-1)/2} \cosh(\nu A_i^R M_i^R) \exp[\nu t_-(M_i) - n \log(M_i)],\\
&\tau_{odd}(t_k, t_{-k}, s) = \\
&\quad \int_0^\infty dM_i \, \frac{\Delta(M)}{\Delta(A)} \prod_{i=1}^n (M_i A_i)^{\nu R + (R-1)/2} \sinh(\nu A_i^R M_i^R) \exp[\nu t_-(M_i) - n \log(M_i)] \ ,
\end{aligned} \tag{3.20}$$

with

$$t_k = -\frac{A^{-k}}{\nu k} \ . \tag{3.21}$$

These expressions are to be thought of as arising from the Fredholm determinants,

$$\tau_{even} = \det(1 + e^{\beta\mu}\tilde{K}_{even-}\tilde{K}_{even+}), \qquad \tau_{odd} = \det(1 + e^{\beta\mu}\tilde{K}_{odd-}\tilde{K}_{odd+}), \tag{3.22}$$

where, for example,

$$\begin{aligned}
[\tilde{K}_{even+}f](x_-) &= \int [dx_+] \cos(e^{-i\beta}x_+x_-) f(x_+),\\
[\tilde{K}_{odd+}f](x_+) &= \int [dx_-] \sin(e^{-i\beta}x_+x_-) f(x_-),
\end{aligned} \tag{3.23}$$

with $[dx_\pm]$ defined in the same way as in (2.10).

To relate these expressions to FZZT branes, observe that the 0B FZZT determinant [36, 44, 50],

$$e^{\pm W(x)} = \det(X(x)^2 - \mu_B)^{\pm 1} = \det\Big((X_+(x) + X_-(x))^2 - \mu_B\Big)^{\pm 1}, \tag{3.24}$$

asymptotes to

$$\det\left(1 - \frac{X_+^2}{\mu'_B}\right)^{\pm 1} = e^{\mp \sum_n \frac{1}{n} \mu_B'^{-n} X_+^{2n}} , \tag{3.25}$$

which, for $R = 1/2$, matches the deformation of the t_k implied by the Baker-Akhiezer function. In a broad sense, we have therefore succeeded in relating a scaling limit of FZZT amplitude to a Kontsevich-like integral expression (3.20). Note however that unlike in the bosonic and the 0A cases, $R = 1/2$ is not the special radius for which (3.20) can be re-expressed in matrix form using the Harish-Chandra/Itzykson-Zuber formula.

Note, also, that the exponentiated macroscopic-loop operator perturbs both the even and the odd sectors simultaneously. As such, it does not probe the two sectors independently. A different object must be introduced to generate purely even or odd deformations. It is not clear what that object is.

To make sense of the integral expression, (3.20), at the non-perturbative level, it might seem that all one has to do is address the issue of contours as we did for the 0A. However, there is an additional subtlety in the 0B case.

To understand the origin of this subtlety, consider turning on a flux. There are two proposals for the theory deformed by a flux. One method is to represent the flux by a relative shift between the chemical potentials of the even and odd sectors. However, it was found that such a prescription gives rise to violation of T-duality at a non-perturbative level [39, 51].

An alternative prescription, where one adjusts the chemical potential of the left and right-moving sectors independently, was proposed in [39]. This prescription leads to partition function that is T-dual at the non-perturbative level. A second advantage of this formalism is that there is a well defined notion of a fermi-sea, which is lost when one deforms the even/odd sectors separately.

Unfortunately, the integrable structure of 0B has only been understood in the case where we consider a flow which acts only on the even or odd sectors separately. In this formalism it is possible to turn on a flux using the first method, in which the even and odd sectors have different chemical potentials, but it seems very difficult to consider the case where the left and right sectors are modified. Moreover, by splitting the fock space into even and

odd sectors, one loses the notion of a single Fermi-sea once any deformations are turned on. This suggests that Toda flow given in [42] may not represent a physical deformation of the background.

Perturbatively, none of this is an issue since the difference between the left/right and even/odd bases only enters non-perturbatively. However, to make the connection between FZZT-branes and Konsevich-like integral expressions at the non-perturbative level, it seems one must first gain a better understanding of the integrable structure of 0B.[10]

4 Conclusions

In this paper, we found a precise relationship between the Imbimbo-Mukhi model and the open string physics living on FZZT branes. The key ingredient in this relationship, which did not play a role in the $c < 1$ theories, was our scaling limit. Allowing the time coordinate of the FZZT brane to be complex, we pushed the location of the brane into the far Minkowski past. This reduced the complicated action of the full FZZT determinant to a pure Toda flow.

Using analytic continuation to reduce a brane amplitude to a purely closed string process is reminiscent of the imaginary brane analysis of [52]. There one does not require a limiting procedure; rotating the brane to imaginary time is all that is required to decouple the open string states. This may well be true in our story; however, the closed string states would, in general, be a mixture of incoming and outgoing states. The extra step of pushing the brane to the far past is probably required to reduce the FZZT brane to a purely incoming source.

We also considered the generalization to $\hat{c} = 1$ models. For type 0A theory, we found that the integral expression generalizing the Imbimbo-Mukhi formula [47, 48] can be related to the scaling limit of a macroscopic-loop operator provided the radius is $R = (1/2)\sqrt{2\alpha'}$. We also provided a prescription to determine the τ-function non-perturbatively along the integrable flow.

A similar extension to 0B turned out to be more subtle. Perturbatively, the computation is straightforward as it reduces to two copies of the $c = 1$ theory. These two copies can be thought of as either the left and right wave functions or, alternatively, the even and odd wave functions.

Picking the even/odd basis, we find that, when $R = (1/2)\sqrt{2\alpha'}$, the FZZT-brane determinant generates the appropriate Toda flow. Unfortunately, the flow is the same for the even and odd sectors. It is not clear if there is a better way to arrange the FZZT-brane so that

[10] Related discussion on integrable structure for minimal 0B models can be found in [44].

it only generates a flow in either the even or odd sectors. Perhaps a more serious problem is that even at $R = (1/2)\sqrt{2\alpha'}$, we do not find that the resulting integral can be rewritten as a matrix integral. This issue arises from the fact that the integral is essentially two copies of the $c = 1$ integral, which required $R = \sqrt{\alpha'}$ to be written in a matrix form.

Non-perturbatively, which basis one chooses matters. In the case when the flux vanishes, the even/odd basis seems most natural as the problem continues to be manifestly factorizable. One can define two commuting Toda flows which generate separately even and odd deformations.

However, when one turns on a flux, using the even/odd basis, one finds that the 0B partition function is no longer T-dual to the partition function of the 0A theory. If, however, one uses the left/right basis of [39], T-duality is restored. Unfortunately, in the left/right basis, it is no longer obvious how one should write down the Toda flow. Thus, to properly generalize the open/closed duality story to the 0B case appears to require a better understanding of the integrable structure of 0B. We hope to address this issue in the near future.

There are a number of interesting open issues. One is to explore open/closed duality for Neumann FZZT branes. Unlike the macroscopic-loop operator for the Dirichlet case [32], the correct realization of the Neumann FZZT brane has not been found. There is an interesting proposal for representing FZZT branes using non-singlet sectors due to Gaiotto [53], but it does not appear to reproduce the correct disk and annulus amplitudes computed using world sheet techniques.

An understanding of the Neumann FZZT branes might shed some light on our issues with the 0B analysis since the T-dual of the Dirichlet FZZT branes in 0B are the Neumann branes in 0A. Unlike in the Dirichlet case, the Neumann brane acts as a source for winding modes. In the matrix model language, winding modes are encoded by the holonomy of the Wilson lines of the gauged quantum mechanics. The fact that there are two gauge groups in the 0A model is related to the existence of the left and right sectors of the 0B theory. Hence, if one can understand the integrable structure of the winding modes of 0A in the presence of flux, it might suggest the correct integrable structure for the 0B side.

As a final point, we mention that the 0A and 0B models are simply two isolated points among a class of non-critical string theories, including affine type 0 theories and the type II theories. These theories are connected by a web of smooth deformations illustrated in [54]. It should be possible to explore the integrable flow and T-duality of each of these theories at the non-perturbative level, and to find a suitable generalization of the Imbimbo-Mukhi integral.

Acknowledgments

We would like to thank T. Eguchi, D. Gaiotto, M. Huang, A. Klemm, J. Maldacena, D. Shih, J. Walcher, X. Yin, and especially N. Seiberg for useful discussions. AH would like to thank the Institute for Advanced Study, the Taiwan National University, and the Sapporo Summer Institute for hospitality where part of this work was done. This work was supported in part by the DOE grant DE-FG02-95ER40896 and funds from the University of Wisconsin.

Appendix A: Conventions for Toda Integrable Structure

In this appendix, we briefly review the machinery of the Toda hierarchy in order to fix our conventions. For a comprehensive introduction to the Toda integrable hierarchy, see [26, 28, 29].

Following the analysis of [23,24],we can the partition function of undeformed $c = 1$ model in terms of the Fredholm determinant,

$$Z(\mu) = \det(1 + e^{-\beta\mu} K_- K_+), \tag{A.1}$$

where

$$\begin{aligned} [K_+ f](x_-) &= \int dx_+ e^{iq^{-1}x_+x_-} f(x_+), \\ [K_- f](x_+) &= \int dx_- e^{-ix_+x_-} f(x_-). \end{aligned} \tag{A.2}$$

Its log is therefore the sum over the eigenvalues of the operator,

$$\log(1 + e^{-\beta\mu} K_- K_+) \ . \tag{A.3}$$

The orthonormal eigenstates, $|E\rangle$, can be expressed in the position basis as

$$\psi^E_{0\pm}(x_\pm) = \langle x_\pm | E\rangle = \frac{1}{\sqrt{2\pi}} e^{\mp i\phi_0(E)/2} x_\pm^{\pm iE - 1/2}, \tag{A.4}$$

and satisfy

$$\langle E | E' \rangle = \delta(E - E') \ . \tag{A.5}$$

One then has

$$\log(Z(\mu)) = \langle E | \log(1 + e^{-\beta(\mu+E)}) | E \rangle = \int dE\, \rho(E) \log(1 + e^{-\beta(\mu+E)}). \tag{A.6}$$

To compute $\rho(E)$, one imposes a boundary condition,

$$\psi^E_+(\Lambda) = \psi^E_-(\Lambda), \tag{A.7}$$

which gives

$$\rho(E) = -\frac{1}{2\pi}\frac{\partial}{\partial E}\phi_0(E) + \frac{1}{\pi}\log\Lambda. \tag{A.8}$$

The $\log(\Lambda)$ term non-universal and can be dropped. By integrating by parts, one arrives at the standard expression,

$$\log(Z(\mu)) = -\frac{\beta}{2\pi}\int dE \frac{\phi_0(E)}{1+e^{\beta(E+\mu)}} \ . \tag{A.9}$$

In the deformed theory, the $\tilde{K}_\pm$ operator is modified;

$$\begin{aligned} [\tilde{K}_+ f](x_-) &= \int [dx_+] e^{iq^{-1}x_+x_-} f(x_+), \\ [\tilde{K}_- f](x_-) &= \int [dx_-] e^{-ix_+x_-} f(x_-) \ . \end{aligned} \tag{A.10}$$

The eigenvectors of the operator $\tilde{K}_-\tilde{K}_+$ can be expressed as

$$\tilde{\psi}^E_\pm(x_\pm) = e^{-iU_\pm(x_\pm)}\psi^E_\pm(x_\pm), \tag{A.11}$$

where

$$\psi^E_\pm(x_\pm) = \langle x_\pm | e^{\pm\frac{i}{2}\phi_0(E)}\hat{\mathcal{W}}_\pm^{-1}|E\rangle. \tag{A.12}$$

The operators $\hat{\mathcal{W}}_\pm$ are called "dressing operators" and take the form,

$$\hat{\mathcal{W}}_\pm = \hat{W}_\pm e^{\sum_{n\geq 1} z_{\pm n}\hat{\omega}^{\mp n/R}}, \qquad \hat{W}_\pm = e^{\pm\frac{1}{2}i(1\pm a)\phi(E)} e^{-iR\sum_{n\geq 1} v_{\pm n}\hat{\omega}^{\pm n/R}}, \tag{A.13}$$

where

$$\hat{\omega} = e^{-i\partial_E} \ . \tag{A.14}$$

They satisfy the relation,

$$\hat{\mathcal{W}}_+ e^{-i\phi_0}\hat{\mathcal{W}}_-^{-1} = 1 \ . \tag{A.15}$$

It is the relation (A.15), which, together with Theorem 1.5 of [26], allows one to relate the computation of the deformed matrix model to the Toda flow. The structure of the Toda flow then allows one to compute $\phi(E)$ and the $v_{\pm n}$'s as a function of $z_{\pm n}$'s, provided one specifies the initial conditions satisfied by the undeformed theory. Once the $\phi(E)$ is found, the partition function can be derived from (A.9).

Since we already know the wave functions which diagonalize K_-K_+ in the undeformed theory, we can write the initial condition for $\hat{\mathcal{W}}_\pm$ in the following form.

$$\hat{\mathcal{W}}_\pm(z_n = 0, z_{-n} = 0) = e^{\pm\frac{1}{2}i(1\pm a)\phi_0(E)} \ . \tag{A.16}$$

Here, the parameter a is introduced to allow for the ambiguity in the overall phase of the wave functions which does not affect any physical results. Indeed, different choices of a give

rise to the same flow [28]. We note this ambiguity here simply to warn the reader of the different conventions found in the references, with $a = 0$ and $a = \pm 1$ being the most popular.

It is customary to parameterize $\hat{W}_\pm$ as

$$\hat{W}_\pm = \sum_{j=0}^{\infty} \hat{w}_j^\pm(E, z_+, z_-)\hat{\omega}^{\pm j}. \tag{A.17}$$

Then, one can identify $\hat{w}_0^\pm$ as being related to $\phi(E)$ in the parameterization of (A.13) according to

$$\hat{w}_0^\pm(E, z_+, z_-) = e^{\pm\frac{i}{2}(1\pm a)\phi(E)} \ . \tag{A.18}$$

This form of $\mathcal{W}$ can be matched to the convention of [26] by identifying

$$\begin{aligned}
\hat{w}_j^{\binom{\infty}{0}}(s, z_+, z_-) &= w_j(E = i(s/R + 1/2), z_+, z_-) \ , \\
\hat{W}^{\binom{\infty}{0}}(z_+, z_-) &= \sum_{j=0}^{\infty} \mathrm{diag}[\hat{w}_j^{\binom{\infty}{0}}]\Lambda^{\mp j} \ , \\
\Lambda &= \hat{\omega}^{-1/R} = e^{\partial_s} \ .
\end{aligned} \tag{A.19}$$

With this choice, $\hat{\mathcal{W}}^{\binom{\infty}{0}}$ takes the form given in (1.2.9) and (1.2.10) of [26]. The sign and phase of $z_{\pm n}$ was chosen in (2.13) in order for the same $z_{\pm n}$ to appear on both side of this identification.

Appendix B: Baker Akhiezer Function

The Baker-Akhiezer function for the Toda lattice hierarchy is defined in terms of the dressing operators [33];

$$\begin{aligned}
\Psi_s^+(\lambda) &= \left(1 + \sum_{n=1}^{\infty} \hat{w}_n^{(\infty)}(s, z_+, z_-)\lambda^{-n/R}\right) \exp(z_+(\lambda^{1/R}) + s\log\lambda^{1/R}), \\
\Psi_s^-(\lambda) &= \left(1 + \sum_{n=1}^{\infty} \hat{w}_n^{(0)}(s, z_+, z_-)\lambda^{n/R}\right) \exp(z_-(\lambda^{-1/R}) + s\log\lambda^{1/R}).
\end{aligned} \tag{B.1}$$

Equivalently, one has

$$\Psi_s^+(\lambda) = \hat{\mathcal{W}}^{(\infty)}\lambda^{s/R}, \qquad \Psi_s^-(\lambda) = \hat{\mathcal{W}}^{(0)}\lambda^{s/R} \ . \tag{B.2}$$

Defining the Lax operator, $L_\pm$, and the Orlov-Shulman operators, $M_\pm$, by

$$L_\pm = \hat{\mathcal{W}}^{\binom{\infty}{0}}\Lambda^{\pm R}\hat{\mathcal{W}}^{\binom{\infty}{0}-1}, \tag{B.3}$$

$$M_\pm = \frac{1}{R}\mathcal{W}^{\left(\begin{smallmatrix}\infty\\0\end{smallmatrix}\right)} s \mathcal{W}^{\left(\begin{smallmatrix}\infty\\0\end{smallmatrix}\right)-1}, \tag{B.4}$$

it follows that the Baker-Akhiezer functions satisfy the differential relations,

$$L_+\Psi_s^+(\lambda) = \lambda\Psi_s^+(\lambda), \qquad L_-\Psi_s^-(\lambda) = \lambda\Psi_s^-(\lambda), \tag{B.5}$$

$$M_+\Psi_s^+(\lambda) = \lambda\frac{\partial}{\partial\lambda}\Psi_s^+(\lambda), \qquad M_-\Psi_s^-(\lambda) = \lambda\frac{\partial}{\partial\lambda}\Psi_s^-(\lambda)\ . \tag{B.6}$$

These Baker-Akhiezer functions are intimately related to the fermion wave functions of the double-scaled matrix model. Specifically,

$$(\psi_\pm^E(x_\pm))^* = \frac{1}{\sqrt{2\pi}} x_\pm^{(1\mp1)/2}\Psi_s^\pm(x_\pm^{\pm1})\Big|_{s=-R(iE+1/2)}. \tag{B.7}$$

Using the formal expression for the Baker-Akhiezer function, (B.2), the orthogonality relation (A.15), and the dictionary (A.19), it follows that

$$\frac{1}{\sqrt{2\pi}}\Psi_s^+(\lambda_+) = \int d\lambda_-^{-1}\, e^{i\lambda_+\lambda_-^{-1}} \frac{1}{\sqrt{2\pi}}\lambda_-\Psi_s^-(\lambda_-). \tag{B.8}$$

The Fourier transform relation for the dual Baker-Akhiezer function (2.34), as well as the Bessel (3.1) and the sine/cosine transform (3.19) for the $\hat{c} = 1$ models, can be derived along similar lines.

Appendix C: The Imbimbo-Mukhi integral for general n

In this section, we show how the properties of the Baker-Akhiezer function determine the $(n+1)\times(n+1)$ Imbimbo-Mukhi integral in terms of the $n \times n$ integral. The computation is essentially the same as the 1×1 case, but the notation is a bit more complicated. It is useful to define

$$\epsilon_n = \sum_{i=1}^n \epsilon(\lambda_{+i}^{-1/R}). \tag{C.1}$$

We then have the relation,

$$\Psi_s^+(-\epsilon_n, z_-, \tilde{\lambda}_+) = \frac{\tau(-\epsilon_n - \epsilon(\lambda^{-1/R}), z_-, s)}{\tau(-\epsilon_n, z_-, s)}\, e^{-\sum_{i,n}\frac{1}{n}\left(\frac{\tilde{\lambda}}{\lambda_i}\right)^{n/R} + s\log(\tilde{\lambda}^{1/R})}, \tag{C.2}$$

so that

$$\tau(-\epsilon(\tilde{\lambda}^{-1/R}) - \epsilon_n, z_-, s) = \Psi_s^+(-\epsilon_n, z_-, \tilde{\lambda})\tau(-\epsilon_n, z_-, s)\, e^{\sum_{i,n}\frac{1}{n}\left(\frac{\tilde{\lambda}}{\lambda_i}\right)^{n/R} - s\log(\tilde{\lambda}^{1/R})}. \tag{C.3}$$

Next, note that

$$\Psi_s^+(-\epsilon_n, z_-, \tilde{\lambda}_+) \tag{C.4}$$
$$= \int d\tilde{\lambda}_-^{-1} e^{i\tilde{\lambda}_+\tilde{\lambda}_-^{-1}} \frac{1}{\sqrt{2\pi}} \Psi_s^-(-\epsilon_n, z_-, \tilde{\lambda}_-)\tilde{\lambda}_-$$
$$= \int d\tilde{\lambda}_-^{-1} e^{i\tilde{\lambda}_+\tilde{\lambda}_-^{-1}} \frac{1}{\sqrt{2\pi}} \frac{\tau(-\epsilon_n, z_- - \epsilon(\tilde{\lambda}_-^{1/R}), s+1)}{\tau(-\epsilon_n, z_-, s)} \exp[z_-(\tilde{\lambda}_-^{-1/R}) + (s+R)\log(\tilde{\lambda}_-^{1/R})].$$

Substituting this into (C.3) gives

$$\tau(-\epsilon_n - \epsilon(\tilde{\lambda}_+^{-1/R}), z_-, s) = e^{\sum_{i,n} \frac{1}{n}\left(\tilde{\lambda}_+/\lambda_i\right)^{n/R} - s\log(\tilde{\lambda}_+^{1/R})}$$
$$\times \int d\tilde{\lambda}_-^{-1} e^{i\tilde{\lambda}_+\tilde{\lambda}_-^{-1}} \frac{1}{\sqrt{2\pi}} \tau(-\epsilon_n, z_- - \epsilon(\tilde{\lambda}_-^{1/R}), s+1) \exp[z_-(\tilde{\lambda}_-^{-1/R}) + (s+R)\log(\tilde{\lambda}_-^{1/R})], \tag{C.5}$$

or, equivalently, using the notation, (2.30), and shifting $n \to n-1$, we find the recursion relation,

$$Z_n(A, t_-, s) = \left(\prod_{i=1}^{n-1} \frac{1}{A_i - A_n}\right) e^{-s\log(A_n) + \sum_{i=1}^{n-1}\log(A_i)} \tag{C.6}$$
$$\times \int \frac{dM_n}{\sqrt{2\pi}} e^{-\nu M_n^R A_n^R} Z_{n-1}(A, t_- - \epsilon(M_n^{-1}), s+1) e^{-\nu t_-(M_n) - (s+1)\log(M_n)}.$$

It is straightforward to check, by direct substitution, that (2.31) solves (C.6).

References

[1] I. R. Klebanov, "String theory in two-dimensions," hep-th/9108019.

[2] R. Dijkgraaf, "Intersection theory, integrable hierarchies and topological field theory," hep-th/9201003.

[3] A. Morozov, "Integrability and matrix models," *Phys. Usp.* **37** (1994) 1–55, hep-th/9303139.

[4] P. H. Ginsparg and G. W. Moore, "Lectures on 2D gravity and 2D string theory," hep-th/9304011.

[5] P. Di Francesco, P. H. Ginsparg, and J. Zinn-Justin, "2D Gravity and random matrices," *Phys. Rept.* **254** (1995) 1–133, hep-th/9306153.

[6] M. Kontsevich, "Intersection theory on the moduli space of curves and the matrix Airy function," *Commun. Math. Phys.* **147** (1992) 1–23.

[7] V. Fateev, A. B. Zamolodchikov, and A. B. Zamolodchikov, "Boundary Liouville field theory. I: Boundary state and boundary two-point function," hep-th/0001012.

[8] J. Teschner, "Remarks on Liouville theory with boundary," hep-th/0009138.

[9] A. B. Zamolodchikov and A. B. Zamolodchikov, "Liouville field theory on a pseudosphere," hep-th/0101152.

[10] J. McGreevy and H. L. Verlinde, "Strings from tachyons: The $c = 1$ matrix reloaded," *JHEP* **12** (2003) 054, hep-th/0304224.

[11] I. R. Klebanov, J. Maldacena, and N. Seiberg, "D-brane decay in two-dimensional string theory," *JHEP* **07** (2003) 045, hep-th/0305159.

[12] N. Seiberg and D. Shih, "Branes, rings and matrix models in minimal (super)string theory," *JHEP* **02** (2004) 021, hep-th/0312170.

[13] E. Witten, "Noncommutative geometry and string field theory," *Nucl. Phys.* **B268** (1986) 253.

[14] D. Gaiotto and L. Rastelli, "A paradigm of open/closed duality: Liouville D-branes and the Kontsevich model," *JHEP* **07** (2005) 053, hep-th/0312196.

[15] S. Giusto and C. Imbimbo, "The Kontsevich connection on the moduli space of FZZT Liouville branes," *Nucl. Phys.* **B704** (2005) 181–207, hep-th/0408216.

[16] J. Maldacena, G. W. Moore, N. Seiberg, and D. Shih, "Exact vs. semiclassical target space of the minimal string," *JHEP* **10** (2004) 020, hep-th/0408039.

[17] A. Hashimoto, M. Huang, A. Klemm, and D. Shih, "Open/closed string duality for topological gravity with matter," *JHEP* **05** (2005) 007, hep-th/0501141.

[18] J. M. Daul, V. A. Kazakov, and I. K. Kostov, "Rational theories of 2D gravity from the two matrix model," *Nucl. Phys.* **B409** (1993) 311–338, hep-th/9303093.

[19] C. Imbimbo and S. Mukhi, "The Topological matrix model of $c = 1$ string," *Nucl. Phys.* **B449** (1995) 553–568, hep-th/9505127.

[20] D. Ghoshal, S. Mukhi, and S. Murthy, "Liouville D-branes in two-dimensional strings and open string field theory," *JHEP* **11** (2004) 027, hep-th/0406106.

[21] S. Y. Alexandrov, V. A. Kazakov, and I. K. Kostov, "2D string theory as normal matrix model," *Nucl. Phys.* **B667** (2003) 90–110, hep-th/0302106.

[22] A. Mukherjee and S. Mukhi, "$c = 1$ matrix models: Equivalences and open-closed string duality," hep-th/0505180.

[23] I. K. Kostov, "Integrable flows in $c = 1$ string theory," *J. Phys.* **A36** (2003) 3153–3172, hep-th/0208034.

[24] I. K. Kostov, "Matrix Quantum Mechanics, String theory, and Toda Lattice Hierarchy," Talk at James H. Simons Workshop on Random Matrix Theory, Stony Brook, Februray 20-23, 2002. http://tonic.physics.sunysb.edu/~verbaarschot/simons/talks/kostov/.

[25] S. Y. Alexandrov, V. A. Kazakov, and I. K. Kostov, "Time-dependent backgrounds of 2D string theory," *Nucl. Phys.* **B640** (2002) 119–144, hep-th/0205079.

[26] K. Takasaki and K. Ueno, "Toda Lattice Hierarchy," *Adv. Stud. Pure Math.* **4** (1984) 1.

[27] R. Dijkgraaf, G. W. Moore, and R. Plesser, "The Partition function of 2D string theory," *Nucl. Phys.* **B394** (1993) 356–382, hep-th/9208031.

[28] T. Takebe, "Toda lattice hierarchy and conservation laws," *Commun. Math. Phys.* **129** (1990) 281–318.

[29] K. Takasaki, *The world of integrable systems.* Kyoritsu, Tokyo, Japan, 2001.

[30] K. Takasaki, "Toda lattice hierarchy and generalized string equations," *Commun. Math. Phys.* **181** (1996) 131–156, hep-th/9506089.

[31] I. K. Kostov, "String equation for string theory on a circle," *Nucl. Phys.* **B624** (2002) 146–162, hep-th/0107247.

[32] G. W. Moore, "Double scaled field theory at $c = 1$," *Nucl. Phys.* **B368** (1992) 557–590.

[33] K. Takasaki and T. Takebe, "Quasiclassical limit of Toda hierarchy and $W(\infty)$ symmetries," *Lett. Math. Phys.* **28** (1993) 165–176, hep-th/9301070.

[34] Harish-Chandra *Am. J. Math.* **79** (1957) 87.

[35] C. Itzykson and J. B. Zuber, "Combinatorics of the modular group. 2. The Kontsevich integrals," *Int. J. Mod. Phys.* **A7** (1992) 5661–5705, hep-th/9201001.

[36] T. Takayanagi and N. Toumbas, "A matrix model dual of type 0B string theory in two dimensions," *JHEP* **07** (2003) 064, hep-th/0307083.

[37] M. R. Douglas *et al.*, "A new hat for the $c = 1$ matrix model," hep-th/0307195.

[38] I. R. Klebanov, J. Maldacena, and N. Seiberg, "Unitary and complex matrix models as 1D type 0 strings," *Commun. Math. Phys.* **252** (2004) 275–323, hep-th/0309168.

[39] J. Maldacena and N. Seiberg, "Flux-vacua in two dimensional string theory," hep-th/0506141.

[40] A. Jevicki and T. Yoneya, "A deformed matrix model and the black hole background in two-dimensional string theory," *Nucl. Phys.* **B411** (1994) 64–96, hep-th/9305109.

[41] K. Demeterfi, I. R. Klebanov, and J. P. Rodrigues, "The exact S matrix of the deformed $c = 1$ matrix model," *Phys. Rev. Lett.* **71** (1993) 3409–3412, hep-th/9308036.

[42] X. Yin, "Matrix models, integrable structures, and T-duality of type 0 string theory," *Nucl. Phys.* **B714** (2005) 137–171, hep-th/0312236.

[43] C. V. Johnson, "Non-perturbative string equations for type 0A," *JHEP* **03** (2004) 041, hep-th/0311129.

[44] N. Seiberg and D. Shih, "Flux vacua and branes of the minimal superstring," *JHEP* **01** (2005) 055, hep-th/0412315.

[45] C. V. Johnson, "Tachyon condensation, open-closed duality, resolvents, and minimal bosonic and type 0 strings," hep-th/0408049.

[46] J. E. Carlisle, C. V. Johnson, and J. S. Pennington, "D-branes and fluxes in supersymmetric quantum mechanics," hep-th/0511002.

[47] H. Ita, H. Nieder, Y. Oz, and T. Sakai, "Topological B-model, matrix models, $\hat{c} = 1$ strings and quiver gauge theories," *JHEP* **05** (2004) 058, hep-th/0403256.

[48] S. Hyun, K. Oh, J. Park, and S. Yi, "Topological B-model and $\hat{c} = 1$ string theory," hep-th/0502075.

[49] L. A. Pando Zayas and D. Vaman, "Condensing momentum modes in 2D 0A string theory with flux," hep-th/0507061.

[50] T. Takayanagi, "Notes on D-branes in 2D type 0 string theory," *JHEP* **05** (2004) 063, hep-th/0402196.

[51] D. J. Gross and J. Walcher, "Non-perturbative RR potentials in the $\hat{c} = 1$ matrix model," *JHEP* **06** (2004) 043, hep-th/0312021.

[52] D. Gaiotto, N. Itzhaki, and L. Rastelli, "Closed strings as imaginary D-branes," *Nucl. Phys.* **B688** (2004) 70–100, hep-th/0304192.

[53] D. Gaiotto, "Long strings condensation and FZZT branes," hep-th/0503215.

[54] N. Seiberg, "Observations on the moduli space of two dimensional string theory," *JHEP* **03** (2005) 010, hep-th/0502156.